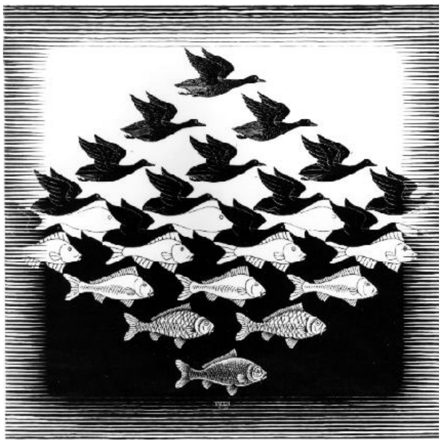
UNIVERSITY OF TORONTO

Faculty of Arts and Science
Ecology and Evolutionary Biology



Mathematical modeling in ecology and evolution

(EEB430/1430)

Midterm Exam – November 3, 2021

Available Time: Two hours

Permitted Materials: Pen/pencil

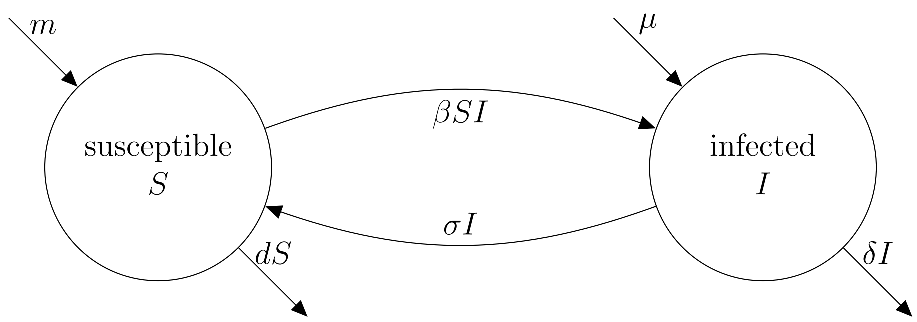
Guidelines:

1. Write your name and student number on the bottom of this page and hand this document at the end of the exam. Your exam will not be graded if you fail to return this question form.
2. The front page lists the materials you can use during this exam. Any other materials are not allowed. This includes mobile devices.
3. When asked, elaborate your answers by providing the equations used or listing the assumptions made. If you need more space for your answers please raise your hand and ask for more paper.
4. Please write clearly.

NB. This is an individual exam. Good luck!

Name:	Student number:

Question 1 [4/23 points] Below is a flow diagram describing the rate of change in the number of individuals who are susceptible to and infected with a pesky virus named 2-VoC-SRAS.



a. [1 points] Explain how a hypothetical new more contagious variant of the virus would affect the parameters.

b. [3 points] Imagine a vaccine is invented that reduces (but does not eliminate) the chance of being infected. Assume that the effect of the vaccine wears off with time and that recovering from infection does not provide protection from future infection. Add a new variable to the flow diagram above to represent this and label the arrows (including new parameters). Describe in words what each added arrow represents.

Question 2 [10/23 points] The model of logistic growth incorporates competition by assuming the number of surviving individuals per parent, R, declines linearly with the size of the population, $R(n) = 1+r(1-n/k)$. This increases as the number of individuals, n, declines to 0. But in some populations offspring production may be impaired when there are too few individuals, e.g., when it is difficult to find mates (this is called an Allee effect). To model an Allee effect we can write the number of surviving individuals per parent as $R(n) = 1 + r(1 - n/k)(n - a)/(k - a)$, with $0 < a < k$. The recursion equation for the size of the population is $n(t + 1) = R(n)n(t)$.

a. [3 points] Find the equilibrium values of n.

b. [6 points] Determine the stability of each equilibrium assuming $r > 0$.

c. [1 points] Assume that a population of woodland caribou follows these dynamics. If this population disappeared from an area and we wanted to re-introduce them, what is the minimum population size we should start with?

Question 3 [9/23 points] Historically, we've typically assumed populations are well adapted to their environments. That is, we've assumed they have trait values that closely match the trait values that optimize fitness. With climate change this may no longer be the case, and an important question is how well populations can evolutionarily track changes in optimal trait values. A simple, and common, model for a population evolving in a changing environment is

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v\gamma(o - z) \tag{1}$$

$$\frac{\mathrm{d}o}{\mathrm{d}t} = k \tag{2}$$

where z is the population mean trait value, o is the optimal trait value, v is the amount of genetic variance in the trait (assumed constant), γ is the strength of selection on the trait (assumed constant), and k is the rate of environmental change (assumed constant).

This is a multivariate model (two variables, z and o), but we can make it univariate by switching our perspective to think about the lag of the mean trait value behind the optimal with a transformation, $L = o - z$.

a. [1 points] Write the differential equation for the change in lag, L, in terms of L alone (i.e., there should be no o or z in this equation).

b. [1 points] What is the equilibrium lag?

c. [2 points] Determine the stability of the equilibrium (assume all parameters are positive).

d. [1 points] Assuming the population goes extinct when the equilibrium lag is greater than some critical value, L_c, what is the fastest rate of environmental change a population can persist under?

e. [3 points] Use separation of variables to find the lag as a function of time, $L(t)$, assuming the population is initially perfectly adapted, $L(0) = 0$. [Hint, the integral of $1/(a - bx)$ with respect to x is $-\ln(a - bx)/b$.]

f. [1 points] Check your answer in (e) by comparing what happens as t goes to infinity to your answer in (b).

——— *End of Exam* ———