# UNIVERSITY OF TORONTO

**Faculty of Arts and Science**
**Department of Ecology and Evolutionary Biology**

# Mathematical modeling in ecology and evolution

(EEB314/1430)

*Midterm Exam – October 9, 2024*

*Available Time: 1h 50m*

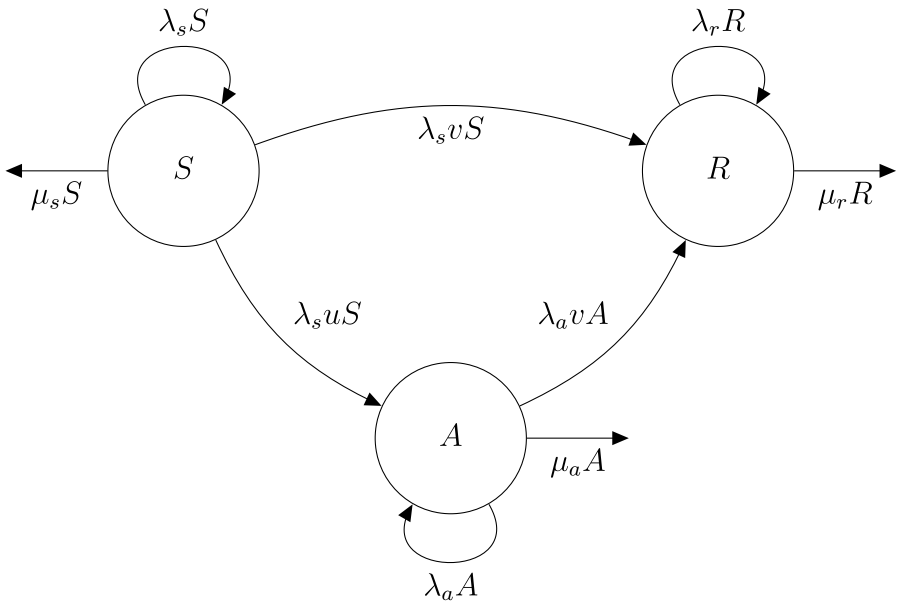
*Permitted Materials: Pen/pencil*

Guidelines:

1. The front page lists the materials you can use during this exam. Any other materials are not allowed. This includes calculators and phones.
2. Write your name and student number on the booklet provided, answer the questions in the booklet, and hand the booklet in at the end of the exam.
3. Please write clearly.

**NB. This is an individual exam.** Good luck!

**Question 1** [ 5/20 points ] Stana *et al.* (2024) study the evolution of drug resistance in cancer. They are particularly concerned about aneuploidy (cells with unusual numbers of chromosomes) contributing to resistance. The issue is that there is a high mutation rate to aneuploidy and aneuploidy may offer partial drug resistance, facilitating full resistance via further mutation. To understand the impact of aneuploid cells on resistance they build a mathematical model describing the number of sensitive cells ($S$), the number of aneuploid cells ($A$), and the number of resistant cells ($R$). They use the following flow diagram to describe their model.




**a.** [ 3 points ] Write down the differential equations corresponding to this flow diagram.

**Answer**

Add the flows coming in and subtract the flows going out

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \lambda_s S - \mu_s S - \lambda_s v S - \lambda_s u S$$
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \lambda_a A - \mu_a A - \lambda_a v A + \lambda_s u S$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \lambda_r R - \mu_r R + \lambda_s v S + \lambda_a v A$$

**b.** [ 2 points ] Here the $\lambda_i$ are cell division rates, the $\mu_i$ are cell death rates, and the $u$ and $v$ are the fraction of daughter cells that mutate. The authors assume $u > v$ and $\mu_a < \mu_s$ in the presence of the drug. What do these two assumptions mean, biologically?

**Answer**

$u > u$ means that sensitive cells produce aneuploid cells at a higher rate than they produce resistant cells. I.e., there is a higher mutation rate to aneuploidy than resistance. $\mu_a < \mu_s$ means that aneuploid cells die at a slower rate than sensitive cells. I.e., aneuploidy offers some drug resistance.

**Question 2** [ 8/20 points ] The Beverton-Holt model (Beverton & Holt 1957),

$$n(t+1) = \frac{Rn(t)}{1+cn(t)},$$

describes density-dependent population growth. Here $R$ is the number of offspring per individual when the population size is small and $c$ describes the impact of competition.

**a.** [ 2 points ] Show that $n(t) = 0$ and $n(t) = (R-1)/c$ are the two equilibria of this model.

**Answer**

We set $n(t+1) = n(t) = \hat{n}$ and solve for $\hat{n}$.

$$\hat{n} = \frac{R\hat{n}}{1+c\hat{n}}$$

This implies $\hat{n} = 0$ or, dividing both sides by $\hat{n}$,

$$\begin{aligned} 1 &= \frac{R}{1+c\hat{n}} \\ 1 + c\hat{n} &= R \\ c\hat{n} &= R - 1 \\ \hat{n} &= \frac{R-1}{c} \end{aligned}$$

**b.** [ 1 points ] When is $n(t) = (R-1)/c$ biologically valid? Assume both $R$ and $c$ are positive.

**Answer**

Because $n(t)$ is a population size, we require $n(t) \geq 0$. This requires $R \geq 1$.

**c.** [ 4 points ] Determine when each equilibrium is stable.

**Answer**

Stability requires the slope of the recursion with respect to the variable evaluated at the equilibrium to have an absolute value less than 1. The slope is

$$\begin{aligned}\frac{\mathrm{d}n(t+1)}{\mathrm{d}n(t)} &= \frac{R(1+cn(t)) - Rn(t)c}{(1+cn(t))^2} \\ &= \frac{R}{(1+cn(t))^2}\end{aligned}$$

Evaluating this at the $\hat{n} = 0$ equilibrium we have

$$\frac{\mathrm{d}n(t+1)}{\mathrm{d}n(t)}|_{n(t)=0} = \frac{R}{(1)^2} = R$$

We therefore need $|R| < 1$ for stability of $\hat{n} = 0$. Evaluating the slope at the $\hat{n} = (R-1)/c$ equilibrium,

$$\frac{\mathrm{d}n(t+1)}{\mathrm{d}n(t)}|_{n(t)=(R-1)/c} = \frac{R}{(1+R-1)^2} = R/R^2 = 1/R$$

We therefore need $|1/R| < 1$ for stability of $\hat{n} = (R-1)/c$. Assuming $R > 0$, we are guaranteed that $-1 < 1/R$ and require only that $1/R < 1$ or, equivalently, $R > 1$. Thus this equilibrium is stable whenever it is biologically valid.

**d.** [ 1 points ] What does the last line in the following Python code do?

```
from sympy import * #import symbolic tools
R,n,c = var('R,n,c') #define parameters and variables
f = R*n/(1 + c*n) #define recursion equation
eqs = solve(f-n,n) #solve for equilibria
fprime = diff(f,n).simplify() #slope of recursion with respect
                                to variable
[fprime.subs(n,i) for i in eqs]
```

**Answer**

The last line evaluates the slope of the recursion at each equilibrium. (We do this because stability requires the slope to be less than 1 in absolute value.)

**Question 3** [ 7/20 points ] Chris Carlson is studying evolution in mutualisms. He is interested in how the mean trait values in two interacting species evolve when an individual has higher fitness when it has a slightly larger trait value than its mutualistic partner. For example, a flower may want to hide its nectar a little deeper than the pollinator's proboscis can reach and the pollinator may want to easily reach the nectar at the bottom of a flower. Let individuals do best when their trait value is $\delta$ larger than the trait value of their partner. And let $\Delta z = \bar{z}_1 - \bar{z}_2$ be the mean trait value of the first species minus the mean trait value of the second species. Then this trait difference evolves like

$$\frac{\mathrm{d}\Delta z}{\mathrm{d}t} = G_1(\delta - \Delta z) - G_2(\Delta z + \delta),$$

where $G_i$ is the amount of genetic variance times the strength of selection in species $i$.

**a.** [ 1 points ] Show that $\Delta z = (G_1 - G_2)\delta/(G_1 + G_2)$ is an equilibrium.

**Answer**

We set the differential equation equal to zero and solve for $\Delta z$,

$$
\begin{aligned}
0 &= G_1(\delta - \Delta z) - G_2(\Delta z + \delta) \\
0 &= (G_1 - G_2)\delta - (G_1 + G_2)\Delta z \\
(G_1 + G_2)\Delta z &= (G_1 - G_2)\delta \\
\Delta z &= (G_1 - G_2)\delta/(G_1 + G_2)
\end{aligned}
$$

**b.** [ 2 points ] Show that this equilibrium is always stable (assuming $G_i > 0$).

**Answer**

We take the derivative of the differential equation with respect to the variable,

$$
\frac{\mathrm{d}}{\mathrm{d}\Delta z}\frac{\mathrm{d}\Delta z}{\mathrm{d}t} = -G_1 - G_2
$$

Given $G_i > 0$, this derivative is always negative and hence any equilibrium is stable.

**c.** [ 3 points ] This is a linear differential equation, so we can find the general solution. The deviation from the equilibrium, $\epsilon(t) = \Delta z(t) - \widehat{\Delta z}$, has solution $\epsilon(t) = \epsilon(0)\exp(-(G_1 + G_2)t)$, where $\widehat{\Delta z}$ is the equilibrium. Use this to derive the general solution for $\Delta z(t)$ in terms of $\Delta z(0)$, $t$, and the parameters.

**Answer**

We write the trait difference in terms of the deviation and sub in the general solution for the deviation, then put everything in terms of trait difference

$$
\begin{aligned}
\Delta z(t) &= \epsilon(t) + \widehat{\Delta z} \\
&= \epsilon(0)\exp(-(G_1 + G_2)t) + \widehat{\Delta z} \\
&= (\Delta z(0) - \widehat{\Delta z})\exp(-(G_1 + G_2)t) + \widehat{\Delta z} \\
&= (\Delta z(0) - \frac{(G_1 - G_2)\delta}{G_1 + G_2})\exp(-(G_1 + G_2)t) + \frac{(G_1 - G_2)\delta}{G_1 + G_2}
\end{aligned}
$$

**d.** [ 1 points ] What must be true about the parameters for species 1 to be "winning" the evolutionary race, i.e., to have a larger mean trait value than species 2, in the long-run?

**Answer**

Species 1 has a larger mean trait value when $\Delta z > 0$. This requires $G_1 > G_2$ in the long-run. (I.e., species 1 must evolve faster for a given distance from its optimum.)

——— *End of Exam* ———