UNIVERSITY OF TORONTO

Faculty of Arts and Science
Department of Ecology and Evolutionary Biology

Mathematical modeling in ecology and evolution

(EEB314/1430)

Test 3 – November 12, 2025

Available Time: 50m

Permitted Materials: Pen/pencil

Question 1 (8/8 points) Many populations face continual environmental change, such as climate warming. Our understanding of how populations will respond to this type of environmental change is facilitated by so-called "moving optimum models". In these models the environmentally-determined phenotype that maximizes fitness, θ, increases linearly at rate c. The mean phenotype of the population, $\bar{z}$, evolves to track this moving optimum, via selection to reduce the mean lag, $\ell = \theta - \bar{z}$. Persistence requires the equilibrium lag to not be too big. Here we'll examine the effect of density-dependent population growth on the predictions of the moving optimum model, following Klausmeier et al. 2020 (Phil. Trans. B).

(a) The simplest scenario is when selection and density-dependence do not interact. This occurs, for example, when birth depends on the mean lag and death depends on the population size, or vice-versa. We consider the former with the following model of population size, n, and mean lag, ℓ,

$$\begin{aligned}\frac{\mathrm{d}n}{\mathrm{d}t} &= n\left(b - \gamma\ell^2/2 - dn\right)\\ \frac{\mathrm{d}\ell}{\mathrm{d}t} &= c - v\frac{\partial}{\partial\ell}\left(\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}t}\right) = c - v\gamma\ell.\end{aligned}$$

Here the mean birth rate, $b - \gamma\ell^2/2$, is reduced by the lag and the death rate, dn, increases with population size. The dynamics of the mean lag, are then independent of population size, increasing due to environmental change, c, and decreasing by evolution, $v\gamma\ell$, where v is the amount of genetic variance in the trait and γ is the strength of selection.

(i) (1 points) Show how to find the equilibrium lag, $\hat{\ell} = c/(v\gamma)$, and the two population size equilibria, $\hat{n} = 0$ and $\hat{n} = (b - c^2/(2v^2\gamma))/d$.

(ii) (2 points) Show how to determine that the second equilibrium, $\hat{\ell} = c/(v\gamma)$ and $\hat{n} = (b - c^2/(2v^2\gamma))/d$, is stable when $\hat{n} > 0$.

(iii) (1 points) Assuming, $\hat{n} = (b - c^2/(2v^2\gamma))/d$, is stable, what is the fastest rate of environmental change that allows long-term persistence? We call this the critical rate of environmental change.

(b) Now consider a scenario where selection and density-dependence interact, e.g., when death depends on both trait value and population size,

$$\begin{aligned}\frac{\mathrm{d}n}{\mathrm{d}t} &= n\left(b - (d + \gamma\ell^2/2)n\right)\\ \frac{\mathrm{d}\ell}{\mathrm{d}t} &= c - v\frac{\partial}{\partial\ell}\left(\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}t}\right) = c - v\gamma\ell n.\end{aligned}$$

(i) (1 points) Show how to determine that at equilibrium, $\hat{\ell} = c/(v\gamma\hat{n})$, with $\hat{n} = (b \pm \sqrt{b^2 - 2dc^2/(v^2\gamma)})/(2d)$.

(ii) (1 points) Explain why, mathematically, these two equilibria are biologically invalid when $b^2v^2\gamma < 2dc^2$.

(iii) (1 points) To determine stability we construct the Jacobian. Evaluating at $\hat{n} = (b + \sqrt{b^2 - 2dc^2/(v^2\gamma)})/(2d)$ we find that the trace and determinant are

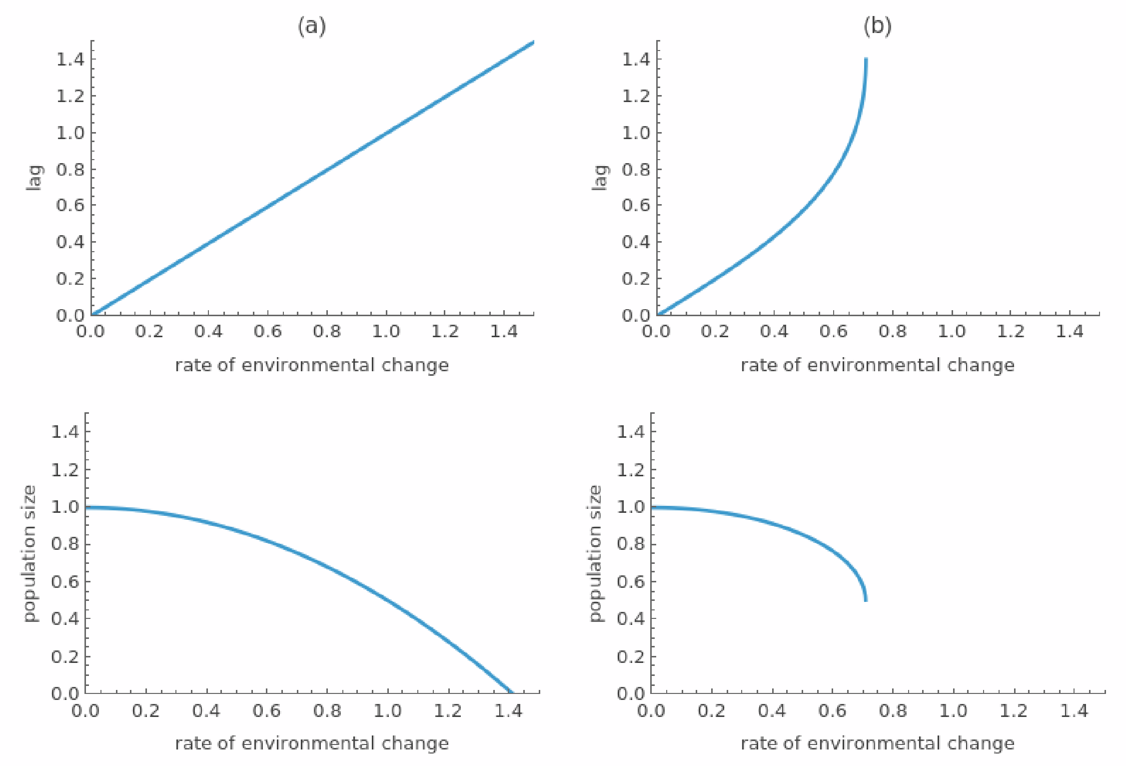
$$\mathrm{Tr}(\mathbf{J}) = -\frac{b(2d + v\gamma) + \sqrt{\gamma(b^2v^2\gamma - 2dc^2)}}{2d}$$

$$\mathrm{Det}(\mathbf{J}) = \frac{b^2v\gamma - 2dc^2/v + \sqrt{\gamma(b^2v^2\gamma - 2dc^2)}}{2d}.$$

Explain why we can conclude that this equilibrium is stable when $b^2v^2\gamma > 2dc^2$. It turns out the other equilibrium is unstable under these conditions.

Note that when $b^2v^2\gamma = 2dc^2$ the equilibrium population size is $\hat{n} = b/(2d) > 0$ but when we increase the rate of environmental change c a little further the equilibria become invalid – biologically, the rate of environmental change $c = bv\sqrt{\gamma/(2d)}$, is a tipping point beyond which populations suddenly go extinct.

(c) (1 points) The stable equilibrium of each model (a and b) is plotted below as a function of the rate of environmental change. Only in the second model (b) do we see a tipping point (in the first model the equilibium is a continuous function of c). Tipping points typically arise when there are positive feedbacks in a system. In 1-3 sentences, explain what the positive feedback in the second model is in biological terms.



End of Exam